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# Evolution Equations Associated with Recursively Defined Bernstein-Type Operators*

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We continue the study of the generalization of Bernstein operators introduced previously, obtained by requiring suitable recursive relations on the binomial-type coefficients. We show that these operators can be used to approximate the solutions of some degenerate second order parabolic problems. © 1996 Academic Press, Inc.

## 1. INTRODUCTION

In [2] we introduced and studied a generalization of the classical Bernstein operators consisting in replacing the binomial coefficients with more general ones satisfying suitable recursive relations. Namely, we considered two sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$ of real numbers and for every $n\geqslant 1$, we defined

$$A_n(f)(x) := \sum_{k=0}^{n} \alpha_{n,k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \qquad f\in\mathscr{C}([0,1]), \quad x\in[0,1], \tag{1.1}$$

with $\alpha_{n,k}$ determined by

$$\alpha_{n+1,k} = \alpha_{n,k} + \alpha_{n,k-1}, \qquad k=1, ..., n \tag{1.2}$$

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and

$$\alpha_{n,0} = \lambda_n, \qquad \alpha_{n,n} = \rho_n. \tag{1.3}$$

In this paper, we require that the sequences $(\lambda_n)_{n \in \mathbb{N}}$ and $(\rho_n)_{n \in \mathbb{N}}$ are positive and converge to $\lambda_\infty$ and $\rho_\infty$ respectively; we also assume that

$$\lambda_\infty > 0, \qquad \rho_\infty > 0. \tag{1.4}$$

Under these assumptions, the function

$$w(x) := \begin{cases} \lambda_\infty, & \text{if} \quad x = 0, \\ \sum_{m=1}^{\infty} \lambda_m x(1-x)^m + \rho_m x^m(1-x), & \text{if} \quad 0 < x < 1, \\ \rho_\infty, & \text{if} \quad x = 1, \end{cases} \tag{1.5}$$

is continuous on $[0, 1]$ and, for every $f \in \mathscr{C}([0, 1])$, the sequence $(A_n(f))_{n \in \mathbb{N}}$ converges uniformly to $w \cdot f$ (see [2, Theorem 2.2]).

Observe that (1.4) guarantees that the function w is strictly positive on $[0, 1]$; really, we shall need only this last condition in the sequel.

Since the sequence $(A_n(\mathbf{1}))_{n \in \mathbb{N}}$ converges to $w$, it is definitively strictly positive and therefore we can consider the operator

$$L_n := \frac{A_n}{A_n(\mathbf{1})}. \tag{1.6}$$

We can also assume that every $L_n$ is a positive contraction. If we assume only that $w$ is strictly positive, this follows by considering sufficiently large values of $n$ for which $A_n$ is positive, which is possible by comparing [2, Theorem 2.4, (2.30) and (2.34)] with (1.1).

The starting point of our investigation is a Voronovskaja-type formula obtained in [2, Theorem 3.4], which can be stated in the following form

$$\lim_{n \to \infty} n(L_n(f)(x) - f(x))$$

$$= \begin{cases} \frac{1}{2} x(1-x) f''(x) + x(1-x) \frac{w'(x)}{w(x)} f(x), & \text{if} \quad 0 < x < 1, \\ 0, & \text{if} \quad x = 0, 1, \end{cases} \tag{1.7}$$

uniformly in $x \in [0, 1]$ for every $f \in \mathscr{C}^2([0, 1])$.

Indeed, property (1.7) provides the link with semigroup theory, whence the connection between these operators and the solutions of some general degenerate second order parabolic problems. Similar problems have been considered for Bernstein operators and other classes of positive operators (for a rather complete exposition of this subject see, e.g., [1, Chapter 6]).

However, all the cases considered in the literature until now regard the approximation of the solutions of parabolic problems having the form

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \alpha(x)\dfrac{\partial^2 u}{\partial x^2}(x,t), \qquad 0<x<1, \quad t>0, \\ \lim\limits_{x\to 0^+} \alpha(x)\dfrac{\partial^2 u}{\partial x^2}(x,t) = \lim\limits_{x\to 1^-} \alpha(x)\dfrac{\partial^2 u}{\partial x^2}(x,t) = 0, \qquad t>0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.8}$$

where $u_0 \in \mathscr{C}([0,1]) \cap \mathscr{C}^2(]0,1[)$ satisfies the Ventcel's boundary conditions

$$\lim_{x\to 0^+} \alpha(x)\, u_0''(x) = \lim_{x\to 1^-} \alpha(x)\, u_0''(x) = 0.$$

For example, in the case of Bernstein operators, the function $\alpha$ is given by $\alpha(x) = x(1-x)/2$. More general situations can be considered using Stancu operators, where $\alpha(x) = bx(1-x)/2$ with $b \geqslant 1$, or Lototsky operators, where $\alpha(x) = \lambda(x)\, x(1-x)/2$ with $\lambda\colon [0,1] \to [0,1]$ continuous and strictly positive (see, e.g., [1, 6.3.4–6.3.8]).

Under assumptions (1.4), we shall see that our operators can be associated to a degenerate parabolic problem having the form

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \alpha(x)\dfrac{\partial^2 u}{\partial x^2}(x,t) + \beta(x)\dfrac{\partial u}{\partial x}(x,t), \qquad 0<x<1, \quad t>0, \\ \lim\limits_{x\to 0^+} \alpha(x)\dfrac{\partial^2 u}{\partial x^2}(x,t) + \beta(x)\dfrac{\partial u}{\partial x}(x,t) \\ \quad = \lim\limits_{x\to 1^-} \alpha(x)\dfrac{\partial^2 u}{\partial x^2}(x,t) + \beta(x)\dfrac{\partial u}{\partial x}(x,t) = 0, \qquad t>0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.9}$$

where $u_0$ satisfies the boundary conditions

$$\lim_{x\to 0^+} \alpha(x)\, u_0''(x) + \beta(x)\, u_0'(x) = \lim_{x\to 1^-} \alpha(x)\, u_0''(x) + \beta(x)\, u_0'(x) = 0, \tag{1.10}$$

and $\alpha(x) = x(1-x)/2$ (as in the case of Bernstein operators). The new fact is the appearance of a "perturbation term" $\beta(x) = x(1-x)\, w'(x)/w(x)$ depending on the sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$.

## 2. EVOLUTION EQUATIONS AND ASSOCIATED SEMIGROUPS

Let $(\lambda_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$ be sequences of positive real numbers satisfying (1.4). In order to solve problem (1.9)–(1.10), we consider the degenerate differential operator $\mathscr{A}: D(\mathscr{A}) \to \mathscr{C}([0,1])$ defined by

$$\mathscr{A}u(x) := \frac{x(1-x)}{2}\, u''(x) + \frac{w'(x)}{w(x)}\, x(1-x)\, u'(x), \qquad 0<x<1, \tag{2.1}$$

and by $\mathscr{A}u(0) = \mathscr{A}u(1) = 0$, on the domain

$$D(\mathscr{A}) := \left\{ u \in \mathscr{C}([0,1]) \cap \mathscr{C}^2(]0,1[) \;\middle|\; \lim_{x\to 0^+} \frac{x(1-x)}{2}\, u''(x) + \frac{w'(x)}{w(x)}\, x(1-x)\, u'(x) = \lim_{x\to 1^-} \frac{x(1-x)}{2}\, u''(x) + \frac{w'(x)}{w(x)}\, x(1-x)\, u'(x) = 0 \right\}. \tag{2.2}$$

Problem (1.9) with conditions (1.10) can be written as follows

$$\begin{cases} \mathcal{u}'(t) = \mathscr{A}\mathcal{u}(t), & t \geqslant 0, \quad \mathcal{u}(t) \in D(\mathscr{A}), \\ \mathcal{u}(0) = u_0, & u_0 \in D(\mathscr{A}), \end{cases} \tag{2.3}$$

where $\mathcal{u}(t) = u(\cdot, t)$ for $t \geqslant 0$.

We note that

$$\mathscr{A}u(x) = \frac{x(1-x)}{2w(x)^2} \frac{d}{dx} (u'(x)\, w(x)^2) \qquad \text{for} \quad 0<x<1$$

and that

$$w'(x) = o\left(\frac{1}{x(1-x)}\right) \qquad \text{as} \quad x\to 0^+, \quad x \to 1^-$$

(see [2, Theorem 3.4]).

Our purpose is to show that the operator $\mathscr{A}$ generates a strongly continuous positive semigroup represented in terms of iterates of the operators $L_n$, $n \geqslant 1$. This, by basic semigroup theory, will furnish the solution of (1.9)–(1.10) as limit of the same iterates.

We begin with the following result.

Proposition 2.1. *The subspace $\mathscr{C}^2([0,1])$ is a core for $\mathscr{A}$.*

*Proof.* We fix $u \in D(\mathscr{A})$ and we show the existence of a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathscr{C}^2([0,1])$ converging to $u$ in the graph norm of $\mathscr{A}$.

First, we construct a sequence $(v_n)_{n\in\mathbb{N}}$ in $D(\mathscr{A})\cap\mathscr{C}^1([0,1])$ converging to $u$ in the graph norm of $\mathscr{A}$ and satisfying $v_n''(x)=O(w'(x))=o(1/x(1-x))$ as $x\to 0^+$ and $x\to 1^-$.

We consider only the interval $[0,1/2]$ since the same argument applies to the interval $[1/2,1]$.

For $n\geqslant 3$ the continuous function $v_n$ is defined by the equations $v_n=u$ in $[1/n,1/2]$ and $(d/dx)(w^2v_n')$ constant in the interval $[0,1/n]$; this constant must be $c_1:=(d/dx)(w^2u')(1/n)$ and hence $w^2v_n'=c_2+c_1(x-1/n)$ on $[0,1/n]$ with $c_2:=(w^2u')(1/n)$. This yields

$$v_n(x)=u(1/n)+\int_{1/n}^{x}\frac{c_1(t-1/n)+c_2}{w(t)^2}\,dt,\qquad x\in[0,1/n].$$

We observe that $v_n\in\mathscr{C}^1([0,1/2])\cap\mathscr{C}^2(]0,1/2])$ and $v_n''(x)=O(w'(x))$ as $x\to 0^+$. Since $w$ is bounded from below, we can write $v_n(x)=u(1/n)+R(n)$ in $[0,1/n]$ with

$$|R(n)|\leqslant C\left(\frac{1}{n^2}\left|u''\left(\frac{1}{n}\right)\right|+\frac{1}{n^2}\left|u'\left(\frac{1}{n}\right)\right|+\frac{1}{n}\left|u'\left(\frac{1}{n}\right)\right|\right),$$

where $C$ is a constant independent of $n$.

Since $u\in D(\mathscr{A})$, we have $\lim_{x\to 0^+}x(d/dx)(u'w^2)=0$. Then, if we consider the function $z(x)=(u'w^2)'$, we obtain $z(x)=o(1/x)$ as $x\to 0^+$ and $u'(x)=w(x)^{-2}\int_{1/2}^{x}z(t)\,dt+c_3$; this yields

$$u'(x)=o(\log x)\qquad\text{as}\quad x\to 0^+$$

and

$$u''(x)=O(w'(x)\,u'(x))=o\left(\frac{\log x}{x}\right)\qquad\text{as}\quad x\to 0^+. \tag{1}$$

Therefore the sequence $(R(n))_{n\in\mathbb{N}}$ converges to 0 whence $(v_n)_{n\in\mathbb{N}}$ converges uniformly to $u$.

Moreover

$$\begin{aligned}\sup_{0\leqslant x\leqslant 1/2}|\mathscr{A}u(x)-\mathscr{A}(v_n)(x)| &= \sup_{0\leqslant x\leqslant 1/n}|\mathscr{A}u(x)-\mathscr{A}(v_n)(x)|\\ &\leqslant \sup_{0\leqslant x\leqslant 1/n}|\mathscr{A}u(x)|+\sup_{0\leqslant x\leqslant 1/n}|\mathscr{A}(v_n)(x)|\end{aligned}$$

$$\leqslant \sup_{0 \leqslant x \leqslant 1/n} |\mathscr{A}u(x)| + \frac{1}{2 \inf_{0 \leqslant x \leqslant 1/2} w(x)^2} \frac{1}{n} \frac{d}{dx}(u'w^2)\left(\frac{1}{n}\right).$$

Since $u \in D(\mathscr{A})$, the last sum converges to 0 as $n \to \infty$, and we conclude that the sequence $(\mathscr{A}(v_n))_{n \in \mathbb{N}}$ converges uniformly to $\mathscr{A}u$ on $[0, 1/2]$.

The second step consists in approximating every function $v_n$ of the preceding sequence with elements of $\mathscr{C}^2([0, 1])$ in the graph norm of $\mathscr{A}$.

So, assume that $v \in \mathscr{C}^1([0, 1]) \cap \mathscr{C}^2(]0, 1[)$ satisfies $v''(x) = O(w'(x))$ as $x \to 0^+$ and $x \to 1^-$.

Also in this case we limit ourselves to the interval $[0, 1/2]$. For every $n \geqslant 3$, in this case we consider the continuous function $u_n$ defined by the relations $u_n = v$ on $[1/n, 1/2]$ and

$$u_n(x) = \frac{1}{2}\left(x - \frac{1}{n}\right)^2 v''\left(\frac{1}{n}\right) + \left(x - \frac{1}{n}\right) v'\left(\frac{1}{n}\right) + v\left(\frac{1}{n}\right), \qquad 0 \leqslant x \leqslant \frac{1}{n}.$$

It is clear that $u_n \in \mathscr{C}^2([0, 1])$. Moreover

$$\sup_{0 \leqslant x \leqslant 1/2} |u_n(x) - v(x)| = \sup_{0 \leqslant x \leqslant 1/n} |u_n(x) - v(x)|$$
$$\leqslant \omega\left(v, \frac{1}{n}\right) + \frac{1}{2n^2}\left|v''\left(\frac{1}{n}\right)\right| + \frac{1}{n}\left|v'\left(\frac{1}{n}\right)\right|$$

and the last sum converges to 0 as $n \to \infty$ since $w'(x) = o(1/x)$ as $x \to 0^+$.

Finally

$$\sup_{0 \leqslant x \leqslant 1/2} |\mathscr{A}(u_n)(x) - \mathscr{A}(v)(x)| \leqslant \sup_{0 \leqslant x \leqslant 1/n} |\mathscr{A}(u_n)(x)| + \sup_{0 \leqslant x \leqslant 1/n} |\mathscr{A}v(x)|.$$

The last sum converges uniformly to 0 since, for every $x \in [0, 1/n]$

$$|\mathscr{A}(u_n)(x)| = \left|\frac{x(1-x)}{2} v''\left(\frac{1}{n}\right) + \frac{w'(x)}{w(x)} x(1-x)\left(\left(x - \frac{1}{n}\right) v''\left(\frac{1}{n}\right) + v'\left(\frac{1}{n}\right)\right)\right|$$
$$\leqslant \frac{1}{n}\left|v''\left(\frac{1}{n}\right)\right| + \sup_{0 \leqslant x \leqslant 1/n} \left|x(1-x)\frac{w'(x)}{w(x)}\right| \left(\frac{1}{n}\left|v''\left(\frac{1}{n}\right)\right| + \left|v'\left(\frac{1}{n}\right)\right|\right)$$

and $v''(x) = o(1/x)$, $w'(x) = o(1/x)$ as $x \to 0^+$. ∎

In the sequel, we denote by $\mathscr{L}: D(\mathscr{L}) \to \mathscr{C}([0,1])$ the operator defined by (see (1.6))

$$\mathscr{L}(f) = \lim_{n \to \infty} n(L_n(f) - f) \tag{2.4}$$

on the domain

$$D(\mathscr{L}) = \{ f \in \mathscr{C}([0,1]) \mid \text{there exists } \lim_{n \to \infty} n(L_n(f) - f) \}. \tag{2.5}$$

Clearly, by (1.7) we have $\mathscr{C}^2([0,1]) \subset D(\mathscr{L})$.
The main theorem is the following.

THEOREM 2.2. *Let $(\lambda_n)_{n \in \mathbb{N}}$ and $(\rho_n)_{n \in \mathbb{N}}$ be sequences of real numbers such that the function w defined by* (1.5) *is strictly positive.*
*Then, the differential operator $\mathscr{A}: D(\mathscr{A}) \to \mathscr{C}([0,1])$ defined by* (2.1) *is the closure of $\mathscr{L}$ and*

$$\mathscr{A} = \mathscr{L} \qquad \textit{on the core } \mathscr{C}^2([0,1]). \tag{2.6}$$

*Moreover, $\mathscr{A}$ is the generator of a strongly continuous semigroup $(T(t))_{t \geqslant 0}$ such that, for every $t \geqslant 0$ and for every sequence $(k(n))_{n \in \mathbb{N}}$ of positive integers satisfying $\lim_{n \to \infty} k(n)/n = t$, we have*

$$T(t) = \lim_{n \to \infty} L_n^{k(n)} \qquad \textit{strongly on } \mathscr{C}([0,1]). \tag{2.7}$$

*Hence, if $u_0 \in D(\mathscr{A})$, the unique solution of* (1.9) *and* (1.10) *is given by*

$$u(x,t) = T(t)\, u_0(x) = \lim_{n \to \infty} L_n^{k(n)}(u_0)(x) \qquad \textit{uniformly in} \quad x \in [0,1]. \tag{2.8}$$

*Proof.* Consider the functions $\alpha: [0,1] \to \mathbb{R}$ and $\beta: ]0,1[ \to \mathbb{R}$ defined by

$$\alpha(x) = \frac{x(1-x)}{2}, \qquad (0 \leqslant x \leqslant 1), \qquad \beta(x) = \frac{w'(x)}{w(x)}\, x(1-x), \qquad (0 < x < 1) \tag{1}$$

and, for $0 < x < 1$, define

$$W(x) := \exp\left( -\int_{1/2}^{x} \frac{\beta(t)}{\alpha(t)}\, dt \right) = \exp\left( -2 \int_{1/2}^{x} \frac{w'(t)}{w(t)}\, dt \right) = \left( \frac{w(1/2)}{w(x)} \right)^2.$$

Since the minimum of $w$ is strictly positive, the function $W$ is integrable over the intervals $[0, 1/2]$ and $[1/2, 1]$. Hence, by a theorem of Clément and Timmermans [3, Theorem 2] the operator $\mathscr{A}$ is closed and is the generator of a strongly continuous semigroup $(S(t))_{t\geqslant 0}$.

Moreover, $\mathscr{C}^2([0,1])$ is a core for $\mathscr{A}$ (see Proposition 2.1) and therefore $(\lambda I-\mathscr{A})(\mathscr{C}^2([0,1]))$ is dense in $\mathscr{C}([0,1])$ for a sufficiently large $\lambda\in\mathbb{R}$.

By Voronovskaja's formula (1.7) we have $\mathscr{L}=\mathscr{A}$ on $\mathscr{C}^2([0,1])$ and consequently $(\lambda I-\mathscr{L})(\mathscr{C}^2([0,1]))=(\lambda I-\mathscr{A})(\mathscr{C}^2([0,1]))$ is dense in $\mathscr{C}([0,1])$. Since every $L_n$ is a positive contraction, we can apply Trotter's theorem [15] (see also [11, Theorem 6.7, p. 96]) and conclude that the closure $\bar{\mathscr{L}}$ of $\mathscr{L}$ generates a strongly continuous positive contraction semigroup $(T(t))_{t\geqslant 0}$ satisfying (2.7).

Finally, $\mathscr{C}^2([0,1])$ is a core also for $\bar{\mathscr{L}}$ and hence $D(\bar{\mathscr{L}})=D(\mathscr{A})$ and $\bar{\mathscr{L}}=\mathscr{A}$; if follows in particular $S(t)=T(t)$ for every $t\geqslant 0$ (see [11, Theorem 1.2.6, p. 6]) and this completes the proof of (2.7) and (2.8). ∎

*Remark.* If we assume

$$|w'(x)|=O\left(\frac{1}{\sqrt{x(1-x)}}\right) \qquad \text{as } x\to 0^+ \text{ and } x\to 1^-,$$

then the function $\beta/\sqrt{\alpha}$ is bounded on $]0,1[$, where

$$\beta(x)=x(1-x)\,\frac{w'(x)}{w(x)} \qquad \text{and} \qquad \alpha(x)=\frac{1}{2}\,x(1-x).$$

Hence we can apply directly a result of Martini (see [9, Theorem 3] and [10]) which implies that the operator $\mathscr{A}$ generates a strongly continuous semigroup on $\mathscr{C}([0,1])$ and that $\mathscr{C}^2([0,1])$ is a core for $\mathscr{A}$. In this case Proposition 2.2 is no longer necessary but suitable assumptions on the sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$ must be taken to guarantee the required behavior at the endpoints.

Now, we investigate further properties of the semigroup generated by the differential operator $\mathscr{A}$. As shown in the following result, in most of the cases its domain does not depend on the function $\beta$.

PROPOSITION 2.3. *Assume that $w$ is strictly positive and*

$$\begin{aligned} w'(x)&=O\left(\frac{1}{x\log x}\right) && \text{as} \quad x\to 0^+,\\ w'(x)&=O\left(\frac{1}{(1-x)\log(1-x)}\right) && \text{as} \quad x\to 1^-. \end{aligned} \tag{2.9}$$

*Then, the domain of the differential operator* $\mathcal{A}$ *defined by* (2.1)–(2.2) *is given by*:

$$D(\mathcal{A}) := \{u \in \mathcal{C}([0,1]) \cap \mathcal{C}^2(]0,1[) \mid \lim_{x\to 0^+} x(1-x)\,u''(x) = \lim_{x\to 1^-} x(1-x)\,u''(x) = 0\}. \tag{2.10}$$

*Proof.* Denote by $G$ the right-hand side of (2.10). If $u \in G$, then $u''(x) = o(1/x)$ as $x \to 0^+$ and this implies $u'(x) = o(\log x)$ as $x \to 0^+$. Hence

$$\lim_{x\to 0^+} \frac{x(1-x)}{2}\,u''(x) + \frac{w'(x)}{w(x)}\,x(1-x)\,u'(x) = 0.$$

The same reasoning applies at the point 1 and therefore $u \in D(\mathcal{A})$.

Conversely, if $u \in D(\mathcal{A})$, as in (1) in the proof of Proposition 2.1, we have $u'(x) = o(\log x)$ as $x \to 0^+$ and hence, by (2.9), $\lim_{x\to 0^+}(w'(x)/w(x))\, x(1-x)\,u'(x) = 0$. By (2.2) it follows $\lim_{x\to 0^+} x(1-x)\,u''(x) = 0$. Applying the same argument at the point 1, we conclude that $u \in G$. ∎

Condition (2.9) requires that the limit function $w$ satisfies a stronger property than strict positivity. In the following discussion we see how this request reflects on the sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$.

Let us assume that

$$w'(x) = O\left(\frac{1}{(1-x)\log(1-x)}\right) \qquad \text{as} \quad x \to 1^-. \tag{2.11}$$

In this case we can write $w(x) = \sum_{k=1}^{\infty} b_k x^k + \varphi(x)$, where $b_k = \rho_k - \rho_{k-1}$ (with the convention $\rho_0 = 0$) and $\varphi$ regular at the point 1.

If $(\rho_m)_{m\in\mathbb{N}}$ is increasing we have $b_k \geqslant 0$ for every $k \geqslant 1$; by evaluating $w'$ at the points $1 - 1/n$, we obtain

$$\sum_{k=1}^{n} k b_k \left(1 - \frac{1}{n}\right)^{k-1} = O\left(\frac{n}{\log n}\right) \qquad \text{as} \quad n \to \infty;$$

since $(1 - 1/n)^{k-1}$ is bounded from below for every $n \in \mathbb{N}$ and $k = 1, ..., n$, the last condition leads to

$$\sum_{k=1}^{n} k b_k = O\left(\frac{n}{\log n}\right) \qquad \text{as} \quad n \to \infty. \tag{2.12}$$

Then condition (2.12) is necessary to ensure (2.11).

In the general case, assume that

$$\sum_{k=1}^{n} k\,|b_k| = O\left(\frac{n}{\log n}\right) \qquad \text{as} \quad n \to \infty. \tag{2.13}$$

Then

$$\log(1-x)\,w'(x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \sum_{k=1}^{\infty} kb_k x^k = -\sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} \frac{kb_k}{n-k}\right) x^n.$$

Put $s_n := \sum_{k=1}^{n-1} kb_k/(n-k)$; by (2.13) we have, for every $N \geqslant 1$,

$$\left|\sum_{n=1}^{N} s_n\right| \leqslant \sum_{k=1}^{N-1} k\,|b_k| \sum_{n=k+1}^{N} \frac{1}{n-k} \leqslant c \sum_{k=1}^{N-1} k\,|b_k| \log N = O(N),$$

and this yields $\log(1-x)\,w'(x) = O(1/(1-x))$ as $x \to 1^-$ (see [13, Section 7.5, p. 225]). Then (2.13) implies the second part of condition (2.9).

In particular, if $(\rho_m)_{m\in\mathbb{N}}$ is increasing, conditions (2.11) and (2.12) are equivalent.

Finally, we observe that a sufficient condition which ensures the validity of (2.13) is given by

$$|b_n| = O\left(\frac{1}{n \log n}\right) \qquad \text{as} \quad n \to \infty. \tag{2.14}$$

In fact

$$\sum_{k=2}^{n} k\,|b_k| = O\left(\sum_{k=2}^{n} \frac{1}{\log k}\right) = O\left(\int_2^n \frac{1}{\log x}\,dx\right) \qquad \text{as} \quad n \to \infty.$$

Putting

$$I_n := \int_2^n \frac{1}{\log x}\,dx \qquad \text{and} \qquad J_n := \int_2^n \frac{1}{\log^2 x}\,dx,$$

we have, integrating by parts,

$$I_n = \frac{n}{\log n} + O(1) + J_n$$

and hence $I_n = O(n/\log n)$ since $\lim_{n\to\infty} I_n/J_n = \infty$.

Conversely, condition (2.14) is also necessary if $(\rho_n)_{n\in\mathbb{N}}$ is increasing and concave. In this case the sequence $(b_k)_{k\in\mathbb{N}}$ is decreasing and converges to 0; so, we have

$$n^2 b_{2n} = O\left(\sum_{k=n+1}^{2n} k b_k\right) = O\left(\frac{n}{\log n}\right) \qquad \text{as} \quad n \to \infty,$$

which implies $b_n = O(1/(n \log n))$ as $n \to \infty$.

We observe that the convergence of the series $\sum_{k=1}^{\infty} b_k$ and condition (2.14) are mutually independent.

Of course, a similar analysis can be carried out for condition (2.9) at the point 0.

## 3. SEQUENCE OF OPERATORS ASSOCIATED TO A DIFFERENTIAL OPERATOR

Starting with a sequence $(A_n)_{n\in\mathbb{N}}$ of recursively defined Bernstein-type operators, in Theorem 2.2 we studied when the corresponding differential operator $\mathscr{A}$ generates a strongly continuous semigroup represented in terms of iterates of these operators.

At this point, we deal with the converse problem of finding sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$ associated to a fixed differential operator. In this way, we recover the operators from the evolution equation.

In the sequel we consider a differential operator $\mathscr{A}: D(\mathscr{A}) \to \mathscr{C}([0, 1])$ having the form

$$\mathscr{A}u(x) := \tfrac{1}{2} x(1-x)\, u''(x) + x(1-x)\, \chi(x)\, u'(x), \qquad 0 < x < 1, \quad (3.1)$$

with $\mathscr{A}u(0) = \mathscr{A}u(1) = 0$, on the domain

$$D(\mathscr{A}) := \left\{ u \in \mathscr{C}([0, 1]) \cap \mathscr{C}(]0, 1[) \;\middle|\; \lim_{x \to 0^+, 1^-} \frac{x(1-x)}{2}\, u''(x) + x(1-x)\, \chi(x)\, u'(x) = 0 \right\} \qquad (3.2)$$

and we ask whether there exist two sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$ of real numbers satisfying (1.4) for which the function $\chi$ can be written as

$$\chi(x) = \frac{w'(x)}{w(x)}, \qquad 0 < x < 1.$$

Indeed, in this case, we can approximate the solution of (2.3) by means of iterates of the operators $L_n$ corresponding to the sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$.

In order to give some examples, we observe that if

$$f(z)=\sum_{n=0}^{\infty} a_n z^n \tag{3.3}$$

has radius of convergence $r\geqslant 1$ and $\sum_{n=0}^{\infty} a_n$ converges, then, by putting, for example, $\lambda_m=a_0$ and $\rho_m=a_0+\cdots+a_m$, $m\geqslant 1$, we can write

$$f(x)=\sum_{m=1}^{\infty} \lambda_m x(1-x)^m+\sum_{m=1}^{\infty} \rho_m(1-x)\,x^m, \qquad 0\leqslant x\leqslant 1; \tag{3.4}$$

therefore, on the interval $[0,1]$, the function $f$ coincides with the function $w$ associated with the sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$ (see (1.5)).

Moreover, if $f$ is strictly positive on $[0,1]$, the sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$ are definitively positive and satisfy (1.4).

EXAMPLE 3.1. Let $\chi$ be a polynomial. By imposing $\chi=w'/w$, we obtain $w(x)=\exp(\int_0^x \chi(t)\,dt)$. Hence in this case $w$ is positive and has the form (3.3).

EXAMPLE 3.2. If $w$ is a polynomial positive in $[0,1]$, the function $\chi(z)=w'(z)/w(z)$ is a complex rational function, real on the real axis, and satisfies $\lim_{|z|\to\infty}\chi(z)=0$; moreover $\chi$ has simple poles with integer positive residuals in $\mathbb{C}\backslash[0,1]$.

Conversely, assume now that $\chi$ is a complex rational function, real on the real axis, satisfying $\lim_{|z|\to\infty}\chi(z)=0$ and having simple poles with positive integer residuals in $\mathbb{C}\backslash[0,1]$. Then, there exist $n, m, n_i, m_i\in\mathbb{N}$ and $x_i\in\mathbb{C}\backslash[0,1]$ such that

$$\chi(z)=\sum_{i=1}^{n}\left(\frac{n_i}{z-z_i}+\frac{n_i}{z-\bar{z}_i}\right)+\sum_{i=1}^{m}\frac{m_i}{z-x_i}$$

and hence $\chi=w'/w$ if

$$w(z)=\pm\prod_{i=1}^{n}(z-z_i)^{n_i}(z-\bar{z}_i)^{n_i}\prod_{i=1}^{m}(z-x_i)^{m_i}.$$

*Remark.* We observe that the case w polynomial is already quite satisfactory from the point of view of approximating the solutions of problem (1.9)–(1.10).

Indeed, if $\chi(x)=w'(x)/w(x)$ ($x\in[0,1]$) with $w\in\mathscr{C}^1([0,1])$ positive, we can consider a sequence $(w_n)_{n\in\mathbb{N}}$ of polynomials satisfying $\lim_{n\to\infty} w_n=w$ and $\lim_{n\to\infty} w'_n=w'$ uniformly on $[0,1]$. For every $n\in\mathbb{N}$, we set $\chi_n:=w'_n(x)/w_n(x)$ and we denote by $\mathscr{A}_n$ the differential operator associated with $w_n$ by (2.1). If $\mathscr{A}$ is defined as in (2.1), we have $\lim_{n\to\infty}\mathscr{A}_n(f)=\mathscr{A}(f)$ for every $f\in\mathscr{C}^2([0,1])$ and moreover $(I-\mathscr{A})(\mathscr{C}^2([0,1]))$ is dense in $\mathscr{C}([0,1])$ (see the proof of Theorem 2.2). As a consequence of the Trotter–Kato theorem (see, e.g., [11, Theorem 4.5, p. 89]), if we denote by $(T(t))_{t\geqslant 0}$ the semigroup generated by $\mathscr{A}$ and by $(T_n(t))_{t\geqslant 0}$ that one generated by $\mathscr{A}_n$, $n\geqslant 1$, we have

$$\lim_{n\to\infty} T_n(t)=T(t) \qquad \text{strongly on } \mathscr{C}([0,1])$$

and the convergence is uniform with respect to $t$ in bounded intervals. This also shows that if $u_0\in D(\mathscr{A})$ $(=D(\mathscr{A}_n)$ for every $n\geqslant 1)$ and if we indicate by $u$ and respectively by $u_n$ the solutions of (1.9)–(1.10) corresponding to the differential operators $\mathscr{A}$ and $\mathscr{A}_n$, then

$$u(x,t)=\lim_{n\to\infty} u_n(x,t) \qquad \text{uniformly in} \quad [0,1]\times[0,t_0]$$

for every $t_0>0$.

A similar discussion can be carried out by considering a sequence of polynomials $(\chi_n)_{n\in\mathbb{N}}$ converging uniformly to $\chi$ (see Example 3.1).

Finally, we consider the general problem of defining sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$ associated to an assigned function $\chi$.

We introduce the set

$$D:=B(0,1)\cap B(1,1) \tag{3.5}$$

and denote by $H(D)$ the space of all holomorphic functions on $D$.

We observe preliminarily that the function $w$ associated to the sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$ is an element of $H(D)$ and consequently, the function $\chi=w'/w$ is meromorphic in $D$, real on the interval $]0,1[$ and has simple poles with positive integer residuals in $D\setminus[0,1]$.

Now, assume that a function $\chi$ is assigned and consider the converse problem of finding a function $w$ such that $\chi=w'/w$. Our result is based on the following considerations.

Let $\Omega$ be a bounded open subset of $\mathbb{C}$; the integral operator

$$Tu(z):=-\frac{1}{\pi}\int_\Omega \frac{u(\zeta)}{\zeta-z}\,dx\,dy, \qquad \zeta=x+iy\in\Omega, \tag{3.6}$$

has the following properties:

(1) $T$ is bounded as an operator from $L^2(\Omega)$ in $L^2(\Omega)$ (see [5, Proposition 3.11, p. 158]);

(2) If $u \in L^\infty(\Omega)$ then $T(u) \in \mathscr{C}(\bar{\Omega})$ (see [5, Proposition 3.13, p. 159]);

(3) If $u \in \mathscr{C}_0^\infty(\Omega)$ then $w = T(u) \in \mathscr{C}^\infty(\mathbb{R}^2)$ and $\partial w/\partial \bar{z} = u$, where $\partial/\partial \bar{z} = \frac{1}{2}((\partial/\partial x) + i(\partial/\partial y))$ (see [6, Theorem 1.2.2, p. 3]).

Now, let $u \in \mathscr{C}_0^\infty(\Omega)$; by property (3), $w = T(u)$ satisfies the first order system

$$\begin{cases} (\mathscr{R}e\, w)_x - (\mathscr{I}m\, w)_y = 2\, \mathscr{R}e\, u; \\ (\mathscr{R}e\, w)_y + (\mathscr{I}m\, w)_x = 2\, \mathscr{I}m\, u. \end{cases}$$

A simple integration by parts yields

$$\int_{B_R} |u|^2 = \frac{1}{4} \int_{B_R} |\nabla w|^2 + \frac{1}{2} \int_{S_R} ((\mathscr{R}e\, w)(\mathscr{I}m\, w)_x\, v_2 - (\mathscr{R}e\, w)(\mathscr{I}m\, w)_y\, v_1)\, d\sigma,$$

where $B_R := \{z \in \mathbb{C} \mid |z| \leqslant R\}$, $S_R = \partial B_R$ and $(v_1, v_2)$ is the exterior normal to $S_R$.

Since $|w(z)| = O(|z|^{-1})$ and $|\nabla w(z)| = O(|z|^{-2})$ for $z \to \infty$, letting $R \to \infty$, the boundary integral tends to zero, whence $\int_{\mathbb{C}} |\nabla w|^2 = 4 \int_\Omega |u|^2$.

This implies, by a density argument, that $T$ is a bounded operator from $L^2(\Omega)$ in $W^{1,2}(\Omega)$. Moreover $\partial T(u)/\partial \bar{z} = u$, in the distribution sense, for every $u \in L^2(\Omega)$.

We point that if $\partial w/\partial \bar{z} = 0$ in the distribution sense, then $w$ is analytic in $\Omega$, since the operator $\partial/\partial \bar{z}$ is analytic-hypoelliptic (see [14, Theorem 3.1, p. 23]).

After these preliminaries, we can state the following

THEOREM 3.3. *Let $\chi$ be a meromorphic function on D, real on the interval* $]0, 1[$ *and with a finite number of simple poles with positive integer residuals in* $D \backslash [0, 1]$. *If $\chi$ admits a continuous extension to $\partial D$, then there exist two converging sequences $(\lambda_n)_{n \in \mathbb{N}}$ and $(\rho_n)_{n \in \mathbb{N}}$ such that the fucntion w defined by* (1.5) *is strictly positive and $\chi = w'/w$.*

*Proof.* We write $\chi = \chi_1 + \chi_2$, where $\chi_1$ and $\chi_2$ have the same properties of $\chi$ and, in addition, $\chi_1 \in \mathscr{C}(\bar{D})$ and $\chi_2$ is a rational function vanishing at infinity. We denote by $p$ the polynomial considered in Example 3.2 satisfying $\chi_2 = p'/p$ and define the function $w$: $\bar{D} \to \mathbb{C}$ by

$$w(z) := p(z) \cdot \exp\left(\int_0^z \chi_1(s)\, ds\right), \qquad z \in \bar{D}.$$

Of course, $\chi = w'/w$ in $D$ and $w$ is real and strictly positive on $[0, 1]$.
Moreover we consider an extension $\tilde{w} \in \mathscr{C}^1(\bar{B}(0, 1))$ and define the function $a \in \mathscr{C}(\bar{\Omega})$ by

$$a(z) := \begin{cases} \dfrac{\partial}{\partial \bar{z}} \tilde{w}(z), & \text{if} \quad z \in \bar{B}(0, 1), \\ 0, & \text{if} \quad z \in \bar{B}(1, 1). \end{cases}$$

Let $\Omega = B(0, 1) \cup B(1, 1)$ and $u = T(a)$; by the above discussion, we have $u \in \mathscr{C}(\bar{\Omega}) \cap W^{1,2}(\Omega)$ and $\partial u/\partial \bar{z} = a$. It follows that $w = \ell + \imath$ on $D$, with $\ell = u$ and $\imath = (\tilde{w} - u)$. Since $\imath$ is analytic in $B(0, 1)$ and $\ell$ is analytic in $B(1, 1)$, we can write

$$\ell(z) = \sum_{n=0}^{\infty} a_n (z-1)^n, \qquad \imath(z) = \sum_{n=0}^{\infty} b_n z^n.$$

Moreover $\ell \in \mathscr{C}(\bar{B}(1,1)) \cap W^{1,2}(B(1,1))$ and $\imath \in \mathscr{C}(\bar{B}(0,1)) \cap W^{1,2}(B(0,1))$, whence $\ell(1 + e^{i\theta})$ is in $\mathscr{C}(\partial(B(1, 1)))$ and its $n$th Fourier coefficients is $a_n$; analogously, $\imath(e^{i\theta})$ is in $\mathscr{C}(\partial(B(0, 1)))$ with $b_n$ as $n$th Fourier coefficient. Since $\ell \in W^{1,2}(B(1, 1))$ and $\imath \in W^{1,2}(B(0, 1))$, we have $\sum_{n=0}^{\infty} n\, |a_n|^2 < +\infty$ and $\sum_{n=0}^{\infty} n\, |b_n|^2 < +\infty$. These conditions imply (see Lemma 3.4 below) $\ell(1 + e^{i\theta}) = \sum_{n=0}^{\infty} a_n (1 + e^{in\theta})$ uniformly on $[-\pi, \pi]$ and $\imath(e^{i\theta}) = \sum_{n=0}^{\infty} b_n e^{in\theta}$ uniformly on $[-\pi, \pi]$. In particular, the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge.

Defining, for every $m \geqslant 1$,

$$\lambda_m := b_0 + a_1 + \cdots + a_m \qquad \text{and} \qquad \rho_m := a_0 + b_1 + \cdots + b_m,$$

we write $w$ in the form (1.5). ∎

For the proof of the announced Lemma 3.4, we denote by $E$ the Banach space of all $f \in \mathscr{C}(\partial B(0, 1))$ such that $\sum_{n \in \mathbb{Z}} n\, |\hat{f}(n)|^2 < +\infty$, endowed with the norm $\|f\|_E = \|f\|_\infty + (\sum_{n \in \mathbb{Z}} n\, |\hat{f}(n)|^2)^{1/2}$. Moreover, $S_n f(\theta) := \sum_{|k| \leqslant n} \hat{f}(k)\, e^{ik\theta}$ will be the $n$th partial sum of the Fourier series of $f$.

Lemma 3.4. *For every $f \in E$, we have $\lim_{n \to \infty} S_n f = f$ in the norm of $E$ and hence uniformly on $[-\pi, \pi]$.*

*Proof.* We consider the $n$th Fejer operator

$$\sigma_n(f)(\theta) := \sum_{|k| \leqslant n-1} \left(1 - \frac{|k|}{n}\right) \hat{f}(k)\, e^{ik\theta}.$$

Then

$$\sigma_n(f)(\theta) = S_{n-1}(f)(\theta) - \frac{1}{n} \sum_{|k| \leqslant n-1} |k|\, \hat{f}(k)\, e^{ik\theta}.$$

Since

$$\left| \sum_{|k| \leqslant n-1} |k| \, \hat{f}(k) \, e^{ik\theta} \right| \leqslant Cn \left( \sum_{|k| \in \mathbb{Z}} |k| \, |\hat{f}(k)|^2 \right)^{1/2},$$

with $C$ independent of $n$ and $\|\sigma_n(f)\|_\infty \leqslant \|f\|_\infty$, we obtain $\|S_n(f)\|_E \leqslant (1+C) \|f\|_E$ whence $\sup_{n \in \mathbb{N}} \|S_n\|_E < +\infty$.

At this point it is sufficient to show that trigonometric polynomials are dense in $E$ and this follows by the inequality

$$\|\sigma_n(f) - f\|_E \leqslant \|\sigma_n(f) - f\|_\infty + \frac{1}{n} \left( \sum_{|k| \leqslant n-1} |k| \, |\hat{f}(k)|^2 \right)^{1/2} + \left( \sum_{|k| \geqslant n} |k| \, |\hat{f}(k)|^2 \right)^{1/2}. \quad \blacksquare$$

## 4. FURTHER QUALITATIVE PROPERTIES AND CONVERGENCE OF ITERATES

In this section, we study some further consequences of Theorem 2.2. We give a partial converse of the uniform Voronovskaja type formula (1.7) and some qualitative properties of the solution of (1.9)–(1.10); a pointwise converse of (1.7) can be investigated with the same methods used in [8, Theorem 4.3] (see also [7]).

Finally, since the solution of (1.9)–(1.10) is expressed in terms of iterates of the operators $L_n$ (see (2.8)), we describe their behavior in some simple cases.

First, consider the differential operator $\mathscr{A}$ defined by (2.1). Under the assumptions (1.4), we have seen in Theorem 2.2 that it coincides with the closure of $\mathscr{L}$ (see (2.4)); since the domain of $\mathscr{A}$ is given by (2.2), (2.5) provides us with the following partial converse of the Voronovskaja type formula.

Proposition 4.1. *Let $(\lambda_n)_{n \in \mathbb{N}}$ and $(\rho_n)_{n \in \mathbb{N}}$ be converging sequences of positive real numbers satisfying* (1.4). *If $f \in \mathscr{C}([0,1])$ and $(n(L_n(f) - f))_{n \in \mathbb{N}}$ is uniformly convergent, then $f \in \mathscr{C}^2(]0,1[)$ and*

$$\lim_{x \to 0^+, 1^-} \frac{x(1-x)}{2} f''(x) + \frac{w'(x)}{w(x)} x(1-x) f'(x) = 0 \tag{4.1}$$

*In particular, if we consider the function* $\gamma$: $[0, 1] \to \mathbb{R}$ *defined by*

$$\gamma(x) = \left(\int_0^1 \frac{1}{w(t)^2}\,dt\right)^{-1} \int_0^x \frac{1}{w(t)^2}\,dt, \tag{4.2}$$

*we have* $\lim_{n\to\infty} n(L_n(f) - f) = 0$ *if and only if*

$$f(x) = f(0) + \gamma(x)(f(1) - f(0)), \qquad x \in [0, 1]. \tag{4.3}$$

Another consequence of Theorem 2.2 concerns the convergence of the iterates of the operators $L_n$, $n \geqslant 1$. As already observed, we have $\mathscr{A}f = 0$ if and only if $f = a\gamma + b$, with $a, b \in \mathbb{R}$.

Some qualitative properties of the function $\gamma$ can be derived directly by [2, Proposition 2.5]. For example, we observe that if $(\lambda_n)_{n\in\mathbb{N}}$ is increasing and $(\rho_n)_{n\in\mathbb{N}}$ is decreasing, then the function $w$ is decreasing (see [2, Proposition 2.5, 3)]) and consequently, since $1/w^2$ is increasing, the function $\gamma$ turns out to be convex.

Moreover, through the substitution $u(x) = v(\gamma(x))$, the differential operator $\mathscr{A}$ can be written as

$$\mathscr{A}u(x) = \frac{1}{2}\,\frac{g(y)(1-g(y))}{w(g(y))^4}\,v_{yy}(y),$$

where $g = \gamma^{-1}$ and $y = \gamma(x)$.

Consequently, the corresponding parabolic problem becomes:

$$\begin{cases} \dfrac{\partial v}{\partial t}(y, t) = \dfrac{1}{2}\,\dfrac{g(y)(1-g(y))}{w(g(y))^4}\,\dfrac{\partial^2 v}{\partial y^2}(y, t), & 0 < y < 1, \quad t > 0, \\ \lim\limits_{y\to 0^+, 1^-} \dfrac{1}{2}\,\dfrac{g(y)(1-g(y))}{w(g(y))^4}\,\dfrac{\partial^2 v}{\partial y^2}(y, t) = 0, & t > 0, \\ v(y, 0) = v_0(y). \end{cases} \tag{4.4}$$

Note that by [4] no regularity assumption on the function $v_0$ is necessary.

It is well known that in this case we have the following properties of the solution $v(y, t)$ (see, e.g., [1, 6.2.7, p. 442]):

(1) $\lim\limits_{t\to\infty} v(y, t) = v_0(0)(1-y) + v_0(1)\,y$ uniformly on $[0, 1]$,

(2) $\lim\limits_{t\to\infty} v(y, t) = 0$ if and only if $v_0(0) = v_0(1) = 0$,

and moreover, if we indicate by $(S(t))_{t \geqslant 0}$ the semigroup generated by the differential operator

$$\mathscr{A}^*(v)(y) = \frac{1}{2} \frac{g(y)(1-g(y))}{w(g(y))^4} v''(y),$$

we also have

$$(3) \qquad \lim_{t \to \infty} (S(t)\, v)(y) = v(0)(1-y) + v(1)\, y$$

$$\text{for every} \quad v \in \mathscr{C}([0,1]), \quad y \in [0,1].$$

Hence, since $\gamma(0)=0$ and $\gamma(1)=1$, we can state the corresponding properties of the solution of problem (1.9)–(1.10), by the change of variable $y = \gamma(x)$.

PROPOSITION 4.2. *Let $(\lambda_n)_{n \in \mathbb{N}}$ and $(\rho_n)_{n \in \mathbb{N}}$ be sequences of positive real numbers such that $\lambda_\infty > 0$ and $\rho_\infty > 0$. Moreover, let $u_0 \in \mathscr{C}([0,1])$ and denote by $u: [0,1] \times \mathbb{R}_+ \to \mathbb{R}$ the unique solution of* (1.9)–(1.10). *Then*

(1) $\lim_{t \to \infty} u(x,t) = u_0(0)(1-\gamma(x)) + u_0(1)\, \gamma(x)$ *uniformly on* $[0,1]$,

(2) $\lim_{t \to \infty} u(x,t) = 0$ *if and only if* $u_0(0) = u_0(1) = 0$.

*Finally, if we define the projection $P: \mathscr{C}([0,1]) \to \mathscr{C}([0,1])$ by*

$$Pu(x) = u(0)(1-\gamma(x)) + u(1)\, \gamma(x), \qquad u \in \mathscr{C}([0,1]), \quad x \in [0,1], \tag{4.5}$$

*we have*

(3) $\lim_{t \to \infty} T(t) = P$ *strongly on* $\mathscr{C}([0,1])$.

This last property (3) together with (2.7) suggests us to investigate the general behavior of the iterates of the operators $L_n$. We begin with the simple case where the integer $n$ is fixed and study the sequence $(L_n^p)_{p \in \mathbb{N}}$. In the sequel we assume that the function $A_n(\mathbf{1})$ is strictly positive. We also consider the subspace

$$F := \{ f \in \mathscr{C}([0,1]) \mid f(0) = f(1) = 0 \} \tag{4.6}$$

and we observe that $L_n(F) \subset F$.

LEMMA 4.3. *For every $f \in F$, we have*

$$\lim_{p \to \infty} L_n^p(f) = 0 \qquad \text{uniformly on } [0, 1]. \tag{4.7}$$

*Proof.* Let $f \in F$, $0 \leqslant f \leqslant 1$; we have

$$0 \leqslant L_n(f)(x) = \frac{\sum_{k=0}^n \alpha_{n,k} x^k (1-x)^{n-k} f(k/n)}{\sum_{k=0}^n \alpha_{n,k} x^k (1-x)^{n-k}}$$
$$\leqslant \frac{\sum_{k=0}^n \alpha_{n,k} x^k (1-x)^{n-k} - (\lambda_n (1-x)^n + \rho_n x^n)}{\sum_{k=0}^n \alpha_{n,k} x^k (1-x)^{n-k}} \leqslant 1 - \delta < 1,$$

where

$$\delta := \min_{0 \leqslant x \leqslant 1} \frac{\lambda_n (1-x)^n + \rho_n x^n}{\sum_{k=0}^n \alpha_{n,k} x^k (1-x)^{n-k}} > 0.$$

Then the norm of $L_n$ as an operator from $F$ into itself is less than 1 and this yields (4.7). ∎

PROPOSITION 4.4. *There exist continuous functions $\gamma_n \in \mathscr{C}([0, 1])$ satisfying $\gamma_n(0) = 0$, $\gamma_n(1) = 1$, $0 \leqslant \gamma_n \leqslant \mathbf{1}$ such that the sequence $(L_n^p)_{p \in \mathbb{N}}$ converges strongly to the projection $P_n : \mathscr{C}([0, 1]) \to \mathscr{C}([0, 1])$ defined by*

$$P_n(u)(x) = u(0)(1 - \gamma_n(x)) + u(1)\, \gamma_n(x), \qquad u \in \mathscr{C}([0, 1]), \quad x \in [0, 1]. \tag{4.8}$$

*Moreover, the projection $P_n$ commutes with $L_n$.*

*Proof.* For every $p \geqslant 1$ and $f \in \mathscr{C}([0, 1])$, we have $L_n^p(f)(0) = f(0)$ and $L_n^p(f)(1) = f(1)$ and therefore $\|L_n^p\| = 1$, hence the spectral radius of $L_n$ is equal to 1. By the positivity of $L_n$, we have $1 \in \sigma(L_n)$ and 1 is a simple pole of the resolvent since $\|L_n\| = 1$.

If $e^{i\theta} \neq 1$ and if $f \in \mathscr{C}([0, 1])$ satisfies $L_n(f) = e^{i\theta} f$, we have $f(0) = L_n(f)(0) = e^{i\theta} f(0)$ and $f(1) = L_n(f)(1) = e^{i\theta} f(1)$, from which $f(0) = 0$ and $f(1) = 0$; so, $f \in F$ and by Lemma 4.3,

$$0 = \lim_{p \to \infty} L_n^p(f) = \lim_{p \to \infty} e^{ip\theta} f;$$

this implies $f = 0$. Hence we have proved that

$$\sigma(L_n) \cap \{z \in \mathbb{C} \mid |z| = 1\} = 1.$$

This yields the strong convergence of $(L_n^p)_{p \in \mathbb{N}}$ to a projection $Q_n$ which commutes with $L_n$ (see, e.g., [12, Theorem 3.1, p. 10]).

Now, put $\gamma_n = Q_n(\mathrm{id})$ and observe that $\mathbf{1}$ and $\gamma_n$ are linearly independent since $\gamma_n(0) = 0$, $\gamma_n(1) = 1$, while $Q_n(\mathbf{1}) = \mathbf{1}$ since $L_n(\mathbf{1}) = \mathbf{1}$. By Lemma 4.3, $Q_n(F) = \{0\}$ and therefore the range of $Q_n$ is generated by $\mathbf{1}$ and $\gamma_n$. Hence $Q_n$ coincides with the projection $P_n$ defined by (4.8) and this completes the proof. ∎

If the integer $n$ is not fixed, we can consider the convergence of the sequence $(L_n^{k(n)})_{n \in \mathbb{N}}$ for a suitable sequence $(k(n))_{n \in \mathbb{N}}$ of positive integers. By (3) of Proposition 4.2 and (2.7), it is natural to ask whether $\lim_{n \to \infty} L_n^{k(n)} = P$ strongly on $\mathscr{C}([0, 1])$ under the assumption $\lim_{n \to \infty} k(n)/n = \infty$. Unfortunately, we are not able to establish this in a complete form; this is equivalent to show that the sequence $(P_n)_{n \in \mathbb{N}}$ converges strongly to $P$ or that the sequence $(\gamma_n)_{n \in \mathbb{N}}$ converges uniformly to $\gamma$ or, again, that the sequence $(\gamma_n)_{n \in \mathbb{N}}$ is equicontinuous.

At the moment, we have only at our disposal the following partial result.

THEOREM 4.5. *If $(k(n))_{n \in \mathbb{N}}$ is a sequence of positive integers satisfying $\lim_{n \to \infty} k(n)/n = \infty$, then $\lim_{n \to \infty} L_n^{k(n)}(f) = 0$ $(= P(f))$ uniformly on $[0, 1]$ for every $f \in F$.*

*Proof.* Let $f \in F$, $\varepsilon > 0$ and $t_0 > 0$ such that $\|T(t) f\| \leqslant \varepsilon$ for every $t \geqslant t_0$. Moreover, consider $v \in \mathbb{N}$ such that, for every $n \geqslant v$, $\|L_n^{[nt_0]}(f) - T(t_0) f\| \leqslant \varepsilon$. Then $\|L_n^{[nt_0]}(f)\| \leqslant 2\varepsilon$ for every $n \geqslant v$; if we take $n \geqslant v$ such that $k_n \geqslant [nt_0]$ we have

$$\|L_n^{k(n)}(f)\| = \|L_n^{k(n)-[nt_0]} L_n^{[nt_0]}(f)\| \leqslant \|L_n^{[nt_0]}(f)\| \leqslant 2\varepsilon. \quad ∎$$

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